# Math 246B Lecture 26 Notes

Daniel Raban

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## 1 Harmonic measures

### 1.1 The Riesz-Herglotz theorem

**Theorem 1.1** (F. Riesz-Herglotz)**.** *$u$ is harmonic in $D$ and*

$$\sup_{0 \leq r < 1} \int_{|z|=1} |u(rz)| \, |dz| \leq C < \infty$$

*if and only if there exists a measure $\mu$ on $\partial D$ such that*

$$u(z) = \int_{|w|=1} P(z, w) \, d\mu(w).$$

*Proof.* Let $u(z) = \int_{|w|=1} P(z, w) \, d\mu(w)$ for $|z| < 1$. Then $u$ is harmonic in $D$, and

$$\begin{aligned} u(re^{it}) &= \int_{[-\pi,\pi)} P(re^{it}, e^{i\varphi}) d\mu(\varphi) \\ &= \int_{[-\pi,\pi)} \frac{1 - r^2}{1 + t^2 - 2r\cos(t - \varphi)} d\mu(\varphi) \\ &= \int_{[-\pi,\pi)} P(re^{i\varphi}, e^{it}) d\mu(\varphi). \end{aligned}$$

So

$$\begin{aligned} \int_{-\pi}^{\pi} |u(re^{it})| \, dt &\leq \int_{-\pi}^{\pi} \left( \int_{[-\pi,\pi)} P(re^{i\varphi}, e^{it}) \, |d\mu(\varphi)| \, dt \right) \\ &= \int_{[-\pi,\pi)} \underbrace{\left( \int_{-\pi}^{\pi} P(re^{i\varphi}, e^{it}) \, dt \right)}_{=2\pi} |d\mu(\varphi)| \\ &\leq 2\pi \int_{[-\pi,\pi)} | \, d\mu(\varphi)|. \end{aligned}$$

Check also that if $u_r(z) = u(rz)$, then for all $\psi \in C(\partial D)$,

$$\frac{1}{2\pi}\int_{|z|=1} u_r(z)\psi(z)\,|dz| \to \int_{|z|=1} \psi(z)\,d\mu(z).$$

The left hand side is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{[-\pi,\pi)} P(re^{it}, e^{i\varphi})\,d\mu(\varphi)\right)\psi(e^{it})\,dt = \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{it}, e^{i\varphi})\psi(e^{it})\,dt\right)\,d\mu(\varphi),$$

where the part in the parentheses on the right is the harmonic extension of $\psi \in C(\overline{D})$, so it converges to $\psi(e^{i\varphi})$ uniformly in $\varphi$ as $r \to 1$. So this goes to $\int_{[-\pi,\pi)} \psi(e^{i\varphi})\,d\mu(\varphi)$.

Conversely, let $u$ be harmonic in $D$ such that

$$\|u_r\||_{L^1(\partial D)} = \int_{-\pi}^{\pi} |u(rz)|\,|dz| \leq C, \qquad 0 \leq r < 1.$$

Here $L^1(\partial D) \subseteq \mathcal{M}(\partial D)$, the space of bounded finite Borel measures on $\partial D$. The space $\mathcal{M}(\partial D)$ is the dual of $C(\partial D)$. By Banach-Alaoglu, there exists a sequence $r_j \to 1$ and a measure $\mu \in \mathcal{M}(\partial D)$ such that $u_{r_j} \to \mu$ weakly: for any $\psi \in C(\partial D)$,

$$\frac{1}{2\pi}\int_{|z|=1} u_{r_j}(z)\psi(z)|dz| \to \int \psi\,d\mu.$$

Finally, for all $j$, $u_{r_j}(z)$ is harmonic near $\overline{D}$, so

$$u(r_j z) = \frac{1}{2\pi}\int_{|w|=1} P(z,w)u(r_j w)\,dz.$$

Letting $j \to \infty$, we get

$$u(z) = \int P(z,w)\,d\mu(w).$$

□

**Remark 1.1.** The measure $\mu$ is unique. Let $h^1 = \{u \in H(D) : \int |u(rz)|\,|dz| \leq C\ \forall r\}$. The theorem says that the **Poisson operator** $\mathcal{P} : \mathcal{M}(\partial D) \to h^1$ is a homeomorphism.

**Corollary 1.1.** *Let $f \in \mathrm{Hol}(D)$ with $\mathrm{Re}(f) \geq 0$. Then there exists a measure $\mu \geq 0$ on $\partial D$ and a constant $c \in \mathbb{R}$ such that*

$$f(z) = ic + \int_{|w|=1} \frac{w+z}{w-z}\,d\mu(w).$$

*Proof.* By the Riesz-Herglotz theorem applied to $\mathrm{Re}(f) \geq 0$, we write

$$\mathrm{Re}(f(z)) = \int_{|w|=1} \mathrm{Re}\left(\frac{w+z}{w-z}\right)\,d\mu(w).$$

So if

$$g(z) = \int_{|w|=1} \frac{w+z}{w-z}\,d\mu(w),$$

then $g \in \mathrm{Hol}(D)$, and $\mathrm{Re}(f) = \mathrm{Re}(f)$. The result follows. □

## 1.2 Boundary behavior of harmonic measures

We would like to understand the boundary behavior of $u \in h^1$.

**Theorem 1.2.** *Let $u \in h^1$, and consider the Lebesgue decomposition of the representing measure $\mu$: $d\mu = f/(2\pi)\,|dz| + d\lambda$, where $f \in L^1(\partial D)$, and $d\lambda$ is singular with respect to $|dz|$.*

1. *Then for a.e. $z \in \partial D$, the radial limit $\lim_{r\to 1} u(rz)$ exists and equals $f(z)$.*
2. *If $d\mu = f/(2\pi)|dz|$ with $f \in L^1$, then $u_r \to f$ in $L^1(\partial D)$.*

We will prove this next time. Here is an application:

**Example 1.1** (Problem 12, Analysis qual, Spring 2016)**.** Let $u$ be real, harmonic in $D$, $u \le M$, and assume that $\lim_{r\to 1} u(rz) \le 0$ for a.e. $z \in \partial D$. Show that $u \le 0$.

Consider $v = M - u \ge 0$, which is harmonic. There exists a measure $\mu \ge 0$ such $v(z) = \int_{|w|=1} P(z,w)\,d\mu(w)$. Writing $d\mu = f/(2\pi)|dz| + d\lambda$, where $f \ge 0$ and $\lambda \ge 0$. By the theorem, $f(z) = \lim_{r\to 1} v(rz) = \lim_{r\to 1}(M - u(rz)) \ge M$. We get

$$v(z) = \underbrace{\int P(z,w)\frac{f}{2\pi}|dw|}_{\ge M} + \underbrace{\int P(z,w)\,d\lambda(w)}_{\ge 0}.$$

So $v \ge M$ in $D$, and we get $u \le 0$ in $D$.